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General considerations

The purpose of this chapter is to explain the general organisation of the book, despite the fact that we hope the handbook is accessible to an unprepared reader. For the customary mathematical notations used throughout the book we refer to the list of notations following the preface.
To scientists novice in the subject of continued fractions we recommend the following order of reading in *Part I* and *Part II*:

- first the *Chapters* 1 through 3 on the fundamental theory of continued fractions,
- then *Chapter* 6, with excursions to *Chapter* 4, on algorithms to construct continued fraction representations,
- and finally the *Chapters* 7 and 8, with *Chapter* 5 as background material, for truncation and round-off error bounds.

0.1 Part one

Part I comprises the necessary theoretic background about continued fractions, when used as a tool to approximate functions. Its concepts and theorems are heavily used later on in the handbook. We deal with three term recurrence relations, linear fractional transformations, equivalence transformations, limit periodicity, continued fraction tails and minimal solutions. The connection between continued fractions and series is worked out in detail, especially the correspondence with formal power series at 0 and ∞.
The continued fraction representations of functions are grouped into several families, the main ones being the S-fractions, C-fractions, P-fractions, J-fractions, T-fractions, M-fractions and Thiele interpolating continued fractions. Most classical convergence results are given, formulated in terms of element and value regions. The connection between C- and P-fractions and Padé approximants on the one hand, and between M-fractions and two-point Padé approximants on the other hand is discussed. To conclude,

several moment problems, their link with Stieltjes integral transform representations and the concept of orthogonality are presented.

0.2 Part two

In *Part II* the reader is offered algorithms to construct different continued fraction representations of functions, known either by one or more formal series representations or by a set of function values. The qd-algorithm constructs C-fractions, the $\alpha\beta$- and FG-algorithms respectively deliver J- and T-fraction representations, and inverse or reciprocal differences serve to construct Thiele interpolating fractions. Also Thiele continued fraction expansions can be obtained as a limiting form.
When evaluating a continued fraction representation, only a finite part of the fraction can be taken into account. Several algorithms exist to compute continued fraction approximants. Each of them can make use of an estimate of the continued fraction tail to improve the convergence. A priori and a posteriori truncation error bounds are developed and accurate round-off error bounds are given.

0.3 Part three

The families of special functions discussed in the separate chapters in *Part III* are the bulk of the handbook and its main goal. We present series and continued fraction representations for several mathematical constants, the elementary functions, functions related to the gamma function, the error function, the exponential integrals, the Bessel functions and also several probability functions. All can be formulated in terms of either hypergeometric or confluent hypergeometric functions. We conclude with a brief discussion of the q-hypergeometric function and its continued fraction representations.
Each chapter in *Part III* is more or less structured in the same way, depending on the availability of the material. We now discuss the general organisation of such a chapter and the conventions adopted in the presentation of the formulas.
All tables and graphs in *Part III* are labelled and preceded by an extensive caption. Detailed information on their use and interpretation is given in the *Sections* 9.2 and 9.3, respectively.

Definitions and elementary properties. The nomenclature of the special functions is not unique. In the first section of each chapter the reader is presented with the different names attached to a single function. The variable z is consistently used to denote a complex argument and x for a real argument.

In a function definition the sign := is used to indicate that the left hand side denotes the function expression at the right hand side, on the domain given in the equation:

$$J(z) := \mathrm{Ln}(\Gamma(z)) - \left(z - \frac{1}{2}\right)\mathrm{Ln}(z) + z - \ln(\sqrt{2\pi}).$$

Here the principal branch of a multivalued complex function is indicated with a capital letter, as in Ln, while the real-valued and multivalued function are indicated with lower case letters, as in ln. The function definition is complemented with symmetry properties, such as mirror, reflection or translation formulas:

$$\mathrm{Ln}(\bar{z}) = \overline{\mathrm{Ln}(z)}.$$

Recurrence relations. Continued fractions are closely related to three-term recurrence relations, also called second order linear difference equations. Hence these are almost omnipresent, as in:

$$\begin{aligned} A_{-1} &:= 1, \qquad A_0 := 0, \\ A_n &:= a_n A_{n-1} + b_n A_{n-2}, \qquad n = 1, 2, 3, \ldots \end{aligned}$$

or

$$\begin{aligned} {}_2F_1(a, b; c+1; z) = &-\frac{c(c-1-(2c-a-b-1)z)}{(c-a)(c-b)z}\,{}_2F_1(a, b; c; z) \\ &- \frac{c(c-1)(z-1)}{(c-a)(c-b)z}\,{}_2F_1(a, b; c-1; z). \end{aligned}$$

The recurrence relations immediately connected to continued fraction theory are listed. Other recurrences may be found in the literature, but may not serve our purpose.

Series expansion. Representations as infinite series are given with the associated domain of convergence. Often these series are power series as in (2.2.2) or (2.2.6). The series in the right hand side and the function in the left hand side coincide, denoted by the equality sign =, on the domain given in the right hand side:

$$\tan(z) = \sum_{k=1}^{\infty} \frac{4^k(4^k-1)|B_{2k}|}{(2k)!} z^{2k-1}, \qquad |z| < \pi/2.$$

Asymptotic series expansion. Asymptotic expansions of the form (2.2.4) or (2.2.8) are given, if available, with the set of arguments where they are valid. Now the equation sign is replaced by the sign $\approx$:

$$J(z) \approx z^{-1} \sum_{k=0}^{\infty} \frac{B_{2k+2}}{(2k+1)(2k+2)} z^{-2k}, \qquad z \to \infty, \quad |\arg z| < \frac{\pi}{2}.$$

Stieltjes transform. For functions that can be represented as Stieltjes integral transforms, or equivalently as convergent S-fractions, positive T-fractions or real J-fractions, specific sharp truncation error estimates exist and the relative round-off error exhibits a slow growth rate when evaluating the continued fraction representation of the function by means of the backward algorithm.
Hence, if possible, the function under consideration or a closely related function is written as a Stieltjes integral transform:

$$\frac{\Gamma(a,z)}{z^a e^{-z}} = \frac{1}{\Gamma(1-a)} \int_0^{\infty} \frac{e^{-t} t^{-a}}{z+t}\, dt, \qquad |\arg z| < \pi, \quad -\infty < a < 1.$$

The conditions on the right hand side of the integral representation, here $|\arg z| < \pi, -\infty < a < 1$, are inherited from the function definition.

S-fraction, regular C-fraction and Padé approximants. S-fraction representations are usually found from the solution of the classical Stieltjes moment problem:

$$e^z E_n(z) = \frac{1/z}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{a_m/z}{1} \right), \qquad a_{2k} = n+k-1, \quad a_{2k+1} = k,$$
$$|\arg z| < \pi, \quad n \in \mathbb{N}.$$

The equality sign $=$ between the left and right hand sides here has to be interpreted in the following way. The convergence of the continued fraction in the right hand side is uniform on compact subsets of the given convergence domain, here $|\arg z| < \pi$, excluding the poles of the function in the left hand side. When the convergence domain of the continued fraction in the right hand side is larger than the domain of the function in the left hand side, it may be regarded as an analytic continuation of that function.
C-fractions can be obtained for instance, by dropping some conditions that ensure the positivity of the coefficients a_m:

$$e^z E_\nu(z) = \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{a_m(\nu) z^{-1}}{1} \right), \qquad |\arg z| < \pi, \quad \nu \in \mathbb{C},$$
$$a_1(\nu) = 1, \quad a_{2j}(\nu) = j + \nu - 1, \quad a_{2j+1}(\nu) = j, \quad j \in \mathbb{N}.$$

A C-fraction is intimately connected with Padé approximants, since its successive approximants equal Padé approximants on a staircase in the Padé table. When available, explicit formulas for the Padé approximants in part or all of the table are given. With the operator $\mathcal{P}_k$ defined as in (15.4.1),

$$r_{m+1,n}(z) = \frac{z^{-1}\mathcal{P}_{m+n}\left({}_2F_0(\nu,1;-z^{-1})\,{}_2F_0(-\nu-m,-n;z^{-1})\right)}{{}_2F_0(-\nu-m,-n;z^{-1})}, \qquad m+1 \geq n.$$

T-fraction, M-fraction and two-point Padé approximants. M-fractions correspond simultaneously to series expansions at 0 and at ∞. For instance, the fraction in the right hand side of

$$\frac{{}_1F_1(a+1;b+1;z)}{{}_1F_1(a;b;z)} = \frac{b}{b-z} + \mathop{\mathrm{K}}_{m=1}^{\infty}\left(\frac{(a+m)z}{b+m-z}\right), \qquad z \in \mathbb{C}, \qquad a \in \mathbb{C}, \quad b \in \mathbb{C} \setminus \mathbb{Z}_0^-$$

corresponds at 0 to the series representation of the function in the left hand side and corresponds at $z = \infty$ to the series representation of

$$-\frac{b}{z}\frac{{}_2F_0(a+1,a-b+1;-1/z)}{{}_2F_0(a,a-b+1;-1/z)}.$$

The two-point Padé approximants $r^{(2)}_{n+k,n-k}(z)$ corresponding to the same series at $z = 0$ and at $z = \infty$, are given by

$$r^{(2)}_{n+k,n-k}(z) = \frac{P_{n-1,k}(\infty,a+1,b,z)}{P_{n,k}(\infty,a,b,z)}, \qquad 0 \leq k \leq n,$$

where

$$\begin{aligned} P_{n,k}(\infty,b,c,z) &:= \lim_{a\to\infty} P_{n,k}(a,b,c,z/a), \qquad 0 \leq k \leq n, \\ &= \mathcal{P}_n({}_1F_1(b;c;z)\,{}_1F_1(-b-n;1-c-k-n;-z)), \end{aligned}$$

for $P_{n,k}(a,b,c,z)$ given by (15.4.9) and the operator $\mathcal{P}_n$ defined in (15.4.1).

Real J-fraction and other continued fractions. Contractions of some continued fractions may result in J-fraction representations. Or minimal solutions of some recurrence relation may lead to yet another continued fraction representation. If closed formulas exist for the partial numerators and denominators of these fractions, these are listed after the usual families of S-, C- and T- or M-fractions. In general, we do not list different equivalent forms of a continued fraction.

Significant digits. Traditionally, the goal in designing mathematical approximations for use in function evaluations or implementations is to minimise the computation time. Our emphasis is on accuracy instead of speed. Therefore our numerical and graphical illustrations essentially focus on the presentation of the number of significant digits achieved by the series and continued fraction approximants. All output is reliable and correctly rounded.
By the presentation of tables and graphs for different approximants, also the speed of convergence in different regions of the complex plane is illustrated. More information on the tables and graphs in this handbook can be found in *Chapter* 9.

Reliability. All series and continued fraction representations in the handbook were verified numerically. So when encountering a slightly different formula from the one given in the original reference, it was corrected because the original work most probably contained a typo.

Further reading

- Similar formula books for different families of functions are [AS64; Ext78; SO87; GR00].
- Books discussing some of the special functions treated in this work are [Luk75; Luk69; AAR99].